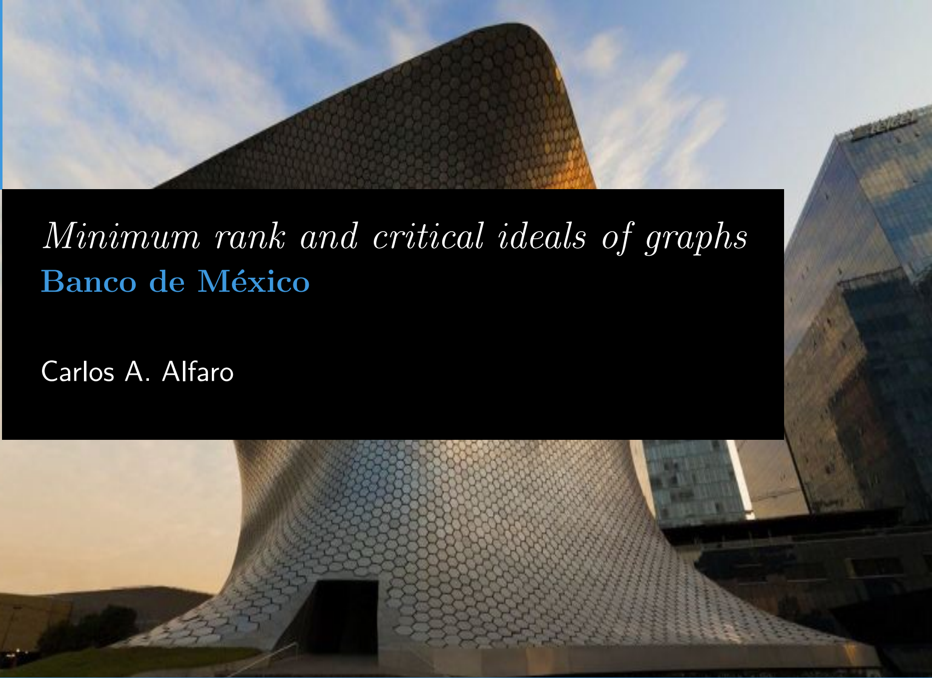

Minimum rank and critical ideals of graphs

Banco de México

Carlos A. Alfaro

Minimum rank

Definition

The minimum rank $\text{mr}_{\mathcal{R}}(G)$ of G is the smallest possible rank among all the $n \times n$ symmetric matrices with entries in the field $\mathcal{R}$, whose u, v-entry $(u \neq v)$ is nonzero whenever u is adjacent to v and zero otherwise.

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Observation

- *Graphs are simple and have no loops,*
- *We might focus when $\mathcal{R}$ is either $\mathbb{R}$ or $\mathbb{C}$.*

Zero-forcing number

Definition

The zero forcing game is a color-change game where vertices can be blue or white. At the beginning, a set of vertices B are colored blue while others remain white. The goal is to color all vertices blue through repeated applications of the color change rule: If u is a blue vertex and v is the only white neighbor of u, then v is forced to become blue. An initial set of blue vertices B is called a zero forcing set if starting with B one can make all vertices blue.

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Definition

The zero forcing number $Z(G)$ is the minimum cardinality of a zero forcing set.

Minimum rank & zero-forcing

Definition

$\text{mz}(G) = |V(G)| - Z(G)$.

Minimum rank & zero-forcing

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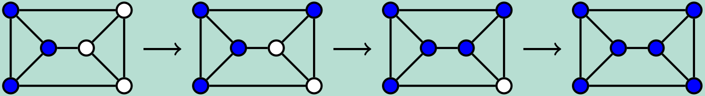
Example



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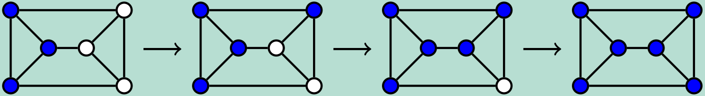

Since there is no zero forcing set of size 2, then $\mathrm{mz}(G) = Z(G) = 3$.

Theorem (AIM Minimum Rank Work Group, 2008)

For every graph G, $\mathrm{mz}(G) \leq \mathrm{mr}_{\mathcal{R}}(G)$ for any field $\mathcal{R}$.

The generalized Laplacian matrix

Let G be a graph with n vertices and $X_G = \{x_u : u \in V(G)\}$ a set of variables.

Definition

The generalized Laplacian matrix $A_X(G)$ of G is the matrix $\mathrm{diag}(X_G) - A(G)$.

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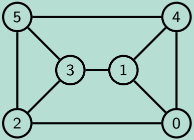


G

$$\begin{bmatrix} x_0 & -1 & -1 & 0 & -1 & 0 \\ -1 & x_1 & 0 & -1 & -1 & 0 \\ -1 & 0 & x_2 & -1 & 0 & -1 \\ 0 & -1 & -1 & x_3 & 0 & -1 \\ -1 & 0 & 0 & 0 & x_4 & -1 \\ 0 & 0 & -1 & -1 & -1 & x_5 \end{bmatrix}$$

Critical ideals of graphs

Let $\mathcal{R}[X_G]$ denote the polynomial ring over a commutative ring $\mathcal{R}$ in the variables X_G.

Definition

For $1 \leq k \leq n$, the k-th critical ideal $I_k^{\mathcal{R}}(G, X_G)$ is the ideal $\langle \mathrm{minors}_k(A_X(G)) \rangle$.

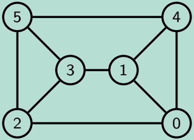
An ideal is said to be trivial if it is equal to $\langle 1 \rangle$ $(= \mathcal{R}[X_G])$.

Definition

The algebraic co-rank $\gamma_{\mathcal{R}}(G)$ of G is the maximum integer k for which $I_k^{\mathcal{R}}(G, X_G)$ is trivial.

Critical ideals of graphs

Example



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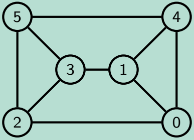
For our graph, $\gamma_{\mathbb{R}}(G) = \gamma_{\mathbb{Z}}(G) = 3$.
And for the first non trivial $I_4^{\mathbb{R}}(G, X_G) = I_4^{\mathbb{Z}}(G, X_G)$, we give the Gröbner basis:

$$\langle x_0 + x_5 - 1, x_1 + x_5 - 1, x_2 - x_5, x_3 - x_5, x_4 + x_5 - 1, x_5^2 - x_5 - 1 \rangle.$$

Critical ideals of graphs

Example



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Note $I_n^{\mathcal{R}}(G, X_G) = \langle \det(A_X(G)) \rangle$.

Varieties of critical ideals of graphs

Definition

The variety $V(I)$ of an ideal I is the set of common roots between polynomials in I.

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Example

The ideal $I_4^{\mathbb{R}}(G, X_G)$ for G of our previous example:
$\langle x_0 + x_5 - 1, x_1 + x_5 - 1, x_2 - x_5, x_3 - x_5, x_4 + x_5 - 1, x_5^2 - x_5 - 1 \rangle$,
there are only two roots in its variety:
$\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right)$
and
$\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right)$.

Varieties of critical ideals of graphs

Example

For the complete graph K_3 with 3 vertices, $\gamma_{\mathbb{R}}(K_3) = 1$,
$I_2^{\mathbb{R}}(K_3, X_{K_3}) = \langle x_0 + 1, x_1 + 1, x_2 + 1 \rangle$, and
$I_3^{\mathbb{R}}(K_3, X_{K_3}) = \langle x_0 x_1 x_2 - x_0 - x_1 - x_2 - 2 \rangle$. The variety
$V(I_2^{\mathbb{R}}(K_3, X_{K_3})) = \{(1,1,1)\}$, and the variety $V(I_3^{\mathbb{R}}(K_3, X_{K_3}))$ is

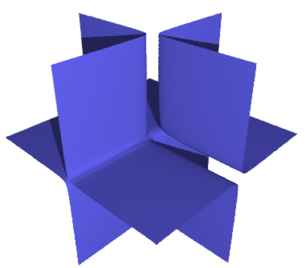

Figure: Partial view of the variety of $I_3^{\mathbb{R}}(K_3, X_{K_3})$ in $\mathbb{R}^3$.

Varieties of critical ideals of graphs

We have that

$$\langle 1 \rangle \supseteq I_1^{\mathcal{R}}(G, X_G) \supseteq \cdots \supseteq I_n^{\mathcal{R}}(G, X_G) \supseteq \langle 0 \rangle.$$

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$$\langle 1\rangle \supseteq I_1^{\mathcal{R}}(G,X_G) \supseteq \cdots \supseteq I_n^{\mathcal{R}}(G,X_G) \supseteq \langle 0\rangle.$$

Then,

$$V(\langle 1\rangle) \subseteq V(I_1^{\mathcal{R}}(G,X_G)) \subseteq \cdots \subseteq V(I_n^{\mathcal{R}}(G,X_G)) \subseteq V(\langle 0\rangle).$$

Minimum rank & critical ideals

Observation (Alfaro & Lin,2019)

If $V(I_k^{\mathcal{R}}(G, X_G)) \neq \emptyset$ for some k, then there exists $\mathbf{a} \in \mathcal{R}$ such that, for all $t \geq k$, $I_t^{\mathcal{R}}(G, \mathbf{a}) = \langle 0 \rangle$; that is, all t-minors of $A_X(G)|_{X_G = \mathbf{a}}$ are equal to 0. Therefore, $\mathrm{mr}_{\mathcal{R}}(G) \leq k - 1$.

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If $V(I_k^{\mathcal{R}}(G, X_G)) \neq \emptyset$ *for some* k*, then there exists* $\mathbf{a} \in \mathcal{R}$ *such that, for all* $t \geq k$*,* $I_t^{\mathcal{R}}(G, \mathbf{a}) = \langle 0 \rangle$*; that is, all t-minors of* $A_X(G)|_{X_G=\mathbf{a}}$ *are equal to* 0*. Therefore,* $\mathrm{mr}_{\mathcal{R}}(G) \leq k-1$*.*

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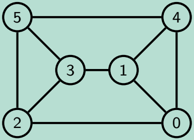

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Since $V(I_4^{\mathbb{R}}(G, X_G))$ is not empty, then $\mathrm{mr}_{\mathbb{R}}(G) \leq 3$.

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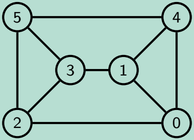

Since $V(I_4^{\mathbb{R}}(G, X_G))$ is not empty, then $\mathrm{mr}_{\mathbb{R}}(G) \leq 3$. Therefore, $\mathrm{mz}(G) = \mathrm{mr}_{\mathbb{R}}(G) = \gamma_{\mathbb{R}}(G) = 3$.

BANCO DE MÉXICO

Minimum rank & critical ideals

Lemma (The Weak Nullstellensatz)

Let $\mathcal{R}$ be an algebraically closed field and let $I \subseteq \mathcal{R}[X]$ be an ideal satisfying $V(I) = \emptyset$. Then I is trivial.

Theorem (Alfaro & Lin, 2019)

If $\mathcal{R}$ is an algebraically closed field, then $\text{mr}_{\mathcal{R}}(G) \leq \gamma_{\mathcal{R}}(G)$.

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Theorem (Alfaro & Lin, 2019)

For every graph G, $\text{mz}(G) \leq \gamma_{\mathcal{R}}(G)$ for any commutative ring $\mathcal{R}$.

Minimum rank & critical ideals

Conjecture

$\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$.

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Minimum rank & critical ideals

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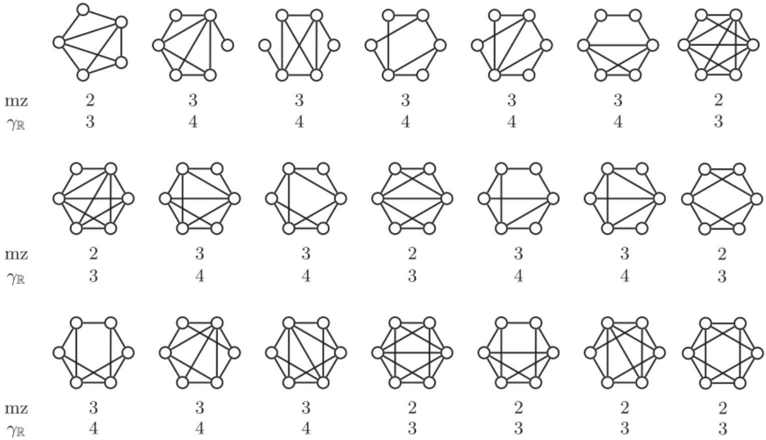
If G is a connected graph with $\text{mr}_{\mathbb{R}}(G) \leq 3$, then $\text{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$.

Computational results

From the 143 connected graphs with at most 6 vertices, only 21 graphs have $\mathrm{mz}(G) < \gamma_{\mathbb{R}}(G)$. For the other graphs, $\mathrm{mz}(G) = \mathrm{mr}(G) = \gamma_{\mathbb{R}}(G)$.



Graphs with equal mz, mr and γ

Theorem (Alfaro & Lin, 2019)

For any tree T, $\mathrm{mz}(T) = \mathrm{mr}(T) = \gamma_{\mathbb{R}}(T)$.

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For any cycle C_n with $n \geq 3$, $\mathrm{mz}(C_n) = \mathrm{mr}(C_n) = \gamma_{\mathbb{R}}(C_n)$.

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For any cycle C_n with $n \geq 3$, $\text{mz}(C_n) = \text{mr}(C_n) = \gamma_{\mathbb{R}}(C_n)$.

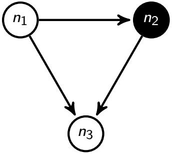
Theorem (Alfaro & Lin, 2019)

Let G be the line graph of a tree. Then $\text{mz}(G) = \text{mr}(G) = \gamma_{\mathbb{R}}(G)$.

Graphs with equal mz, mr and γ

Theorem (Alfaro, Valencia & Vazquez, 2018)

Let D be a connected digraph. Then, $\gamma_{\mathbb{Z}}(D) \leq 1$ if and only if D is isomorphic to Λ_{n_1,n_2,n_3}.



Graphs with equal mz, mr and γ

Theorem (Alfaro & Lin, 2019)

Let $\mathcal{R}$ be a commutative ring with unity. The following are equivalent:

1. D *is isomorphic to* Λ_{n_1,n_2,n_3},
2. $\text{mr}_{\mathcal{R}}(D) \leq 1$,
3. $\text{mz}(D) \leq 1$,
4. $\gamma_{\mathcal{R}}(D) \leq 1$.

Main references

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Thank you!

Carlos A. Alfaro
carlos.alfaro@banxico.org.mx