The degree-distance and transmission-adjacency matrices

25th Conference of the ILAS 2023

Carlos A. Alfaro (BANXICO)

joint work with Octavio Zapata (UNAM)

- To understand to what extent graphs are characterized by their spectra.
- motivated by the graph isomorphism problem

Cospectral graphs

M-cospectral graphs are graphs that share M-spectrum.
Where M might be A, L, Q, D, D^L and D^Q.

Let $\mathcal{G}_n$ be the set of **connected** graphs with n vertices,
and $\mathcal{G}_n^{sp}(M)$ be the set of graphs in $\mathcal{G}_n$ with a M-cospectral mate.

n	4	5	6	7	8	9	10
$\lvert\mathcal{G}_n\rvert$	6	21	112	853	11,117	261,080	11,716,571
$\lvert\mathcal{G}_n^{sp}(A)\rvert$	0	0	2	63	1,353	46,930	2,462,141
$\lvert\mathcal{G}_n^{sp}(L)\rvert$	0	0	4	115	1,611	40,560	1,367,215
$\lvert\mathcal{G}_n^{sp}(Q)\rvert$	0	2	10	80	1,047	17,627	615,919
$\lvert\mathcal{G}_n^{sp}(D)\rvert$	0	0	0	22	658	25,058	1,389,986
$\lvert\mathcal{G}_n^{sp}(D^L)\rvert$	0	0	0	43	745	20,455	787,851
$\lvert\mathcal{G}_n^{sp}(D^Q)\rvert$	0	2	6	38	453	8,168	319,324

Table: Number of connected graphs with an M-cospectral mate.

Coinvariant graphs

- Two matrices M, N are *equivalent* if there exist unimodular matrices P and Q with entries in $\mathbb{Z}$ satisfying $M = PNQ$.
- The Smith normal form of a integer matrix M, denoted by SNF(M), is the unique diagonal matrix $\text{diag}(f_1, \ldots, f_r, 0, \ldots, 0)$ equivalent to M such that $r = rank(M)$ and $f_i | f_j$ for $i < j$.
- The *invariant factors* (or *elementary divisors*) of M are the integers in the diagonal of the SNF(M).

Example

$$L(K_4) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \qquad \text{SNF}(L(K_4)) = \text{diag}(1, 4, 4, 0)$$

Coinvariant graphs

Two graphs G and H are *M-coinvariant* if the SNFs of $M(G)$ and $M(H)$, computed over $\mathbb{Z}$, are the same.

$\mathcal{G}_n^{in}(M)$ is the set of graphs in $\mathcal{G}_n$ with a M-coinvariant mate.

n	4	5	6	7	8	9	10
$\lvert\mathcal{G}_n\rvert$	6	21	112	853	11,117	261,080	11,716,571
$\lvert\mathcal{G}_n^{in}(A)\rvert$	4	20	112	853	11,117	261,080	11,716,571
$\lvert\mathcal{G}_n^{in}(L)\rvert$	2	8	57	526	8,027	221,834	11,036,261
$\lvert\mathcal{G}_n^{in}(Q)\rvert$	2	11	78	620	7,962	201,282	10,086,812
$\lvert\mathcal{G}_n^{in}(D)\rvert$	2	15	102	835	11,080	260,991	11,716,249
$\lvert\mathcal{G}_n^{in}(D^L)\rvert$	0	0	0	18	455	16,505	642,002
$\lvert\mathcal{G}_n^{in}(D^Q)\rvert$	0	2	4	20	259	7,444	264,955

Table: Number of connected graphs with an M-cospectral mate.

Cospectral vs coinvariant

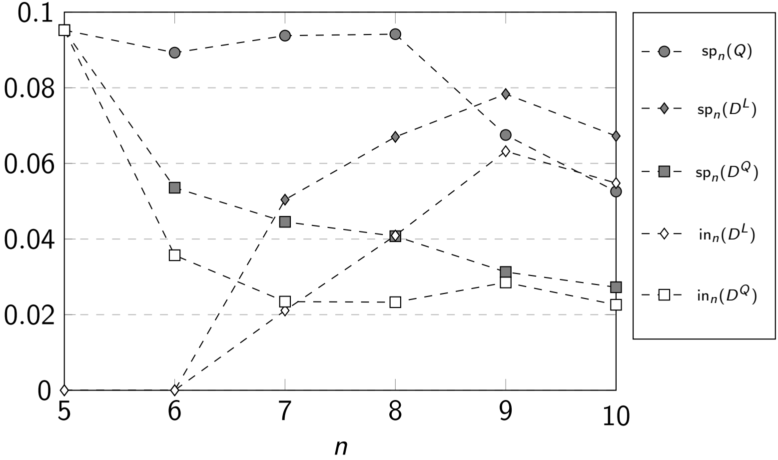

Figure: The fraction of connected graphs on n vertices having a M-cospectral mate is denoted as *sp*. The fraction of connected graphs on n vertices having a M-coinvariant mate is denoted as *in*.

deg $-D$ and trs $-A$ matrices

Let $\deg(G)$ denote the diagonal matrix with the degrees of the vertices of G in the diagonal.

The *transmission* $\mathrm{trs}(u)$ of vertex u is $\sum_{v\in V(G)} \mathrm{dist}(u,v)$.
Let $\mathrm{trs}(G)$ denote the diagonal matrix with the transmissions of the vertices of G in the diagonal.

- the *degree-distance* matrix $D^{\deg}(G)$ of G as $\deg(G) - D(G)$,
- the *transmission-adjacency* matrix $A^{\mathrm{trs}}(G)$ of G as $\mathrm{trs}(G) - A(G)$,
- the *signless degree-distance* matrix $D_{+}^{\deg}(G)$ of G as $\deg(G) + D(G)$, and
- the *signless transmission-adjacency* matrix $A_{+}^{\mathrm{trs}}(G)$ of G as $\mathrm{trs}(G) + A(G)$.

Enumeration

n	4	5	6	7	8	9	10
$\lvert\mathcal{G}_n\rvert$	6	21	112	853	11,117	261,080	11,716,571
$\lvert\mathcal{G}_n^{sp}(D^{\text{deg}})\rvert$	0	2	6	40	485	9,784	355,771
$\lvert\mathcal{G}_n^{sp}(D_+^{\text{deg}})\rvert$	0	0	0	61	901	24,095	852,504
$\lvert\mathcal{G}_n^{sp}(A^{\text{trs}})\rvert$	0	2	6	38	413	7,877	299,931
$\lvert\mathcal{G}_n^{sp}(A_+^{\text{trs}})\rvert$	0	0	0	43	728	19,757	765,421
$\lvert\mathcal{G}_n^{in}(D^{\text{deg}})\rvert$	2	2	6	34	538	17,497	902,773
$\lvert\mathcal{G}_n^{in}(D_+^{\text{deg}})\rvert$	2	11	46	495	7,169	209,822	10,815,879
$\lvert\mathcal{G}_n^{in}(A^{\text{trs}})\rvert$	0	2	4	22	240	6,642	237,118
$\lvert\mathcal{G}_n^{in}(A_+^{\text{trs}})\rvert$	0	0	0	16	456	15,952	605,625

Table: Number of connected graphs with an M-cospectral mate and with an M-coinvariant mate for the matrices $A^{\text{trs}}(G)$, $A_+^{\text{trs}}(G)$, $D_+^{\text{deg}}(G)$ and $D^{\text{deg}}(G)$.

Classic vs new

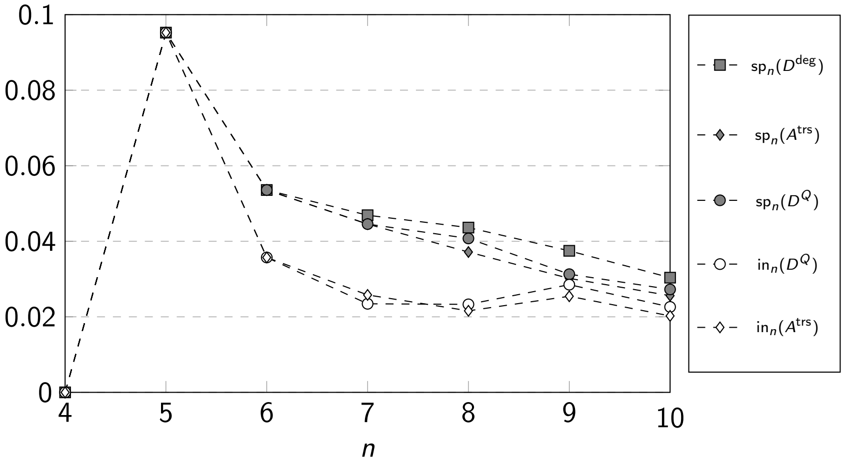

Figure: The parameters $sp_n(M)$ and $in_n(M)$ represent the fraction of graphs with n vertices that have at least one M-cospectral or M-coinvariant mate, respectively. We only show the five best performing parameters.

Cospectral and coinvariant trees

There are no D_+^{deg}-cospectral trees nor D^{deg}-cospectral trees nor A^{trs}-cospectral trees with up to 20 vertices.
There are no A_+^{trs}-coinvariant trees nor A^{trs}-coinvariant trees with up to 20 vertices.

Conjecture

- *Trees are determined by spectrum of the matrices* D_+^{deg}, D^{deg} *and* A^{trs}, *and*
- *Trees are determined by the SNF of the matrices* A_+^{trs} *and* A^{trs}.

Graphs determined by SNF

Theorem

Complete graphs are determined by the SNF of the matrices A^{trs}, A^{trs}_{+}, D^{deg} *and* D^{deg}_{+}.

Future research

- To explore relations between combinatorial properties of graphs with spectrum and SNF of deg-dist and trs-adj matrices.
    - A^{trs} is related with L
- to explore variations of deg-dist and trs-adj matrices.
    - generalized spectrum and generalized SNF (complements).
    - normalized versions

Main references

- A. Abiad & C.A. Alfaro, **Enumeration of cospectral and coinvariant graphs.** *Appl. Math. Comput.* 408 (2021) 126348.
- A. Abiad, C.A. Alfaro & R.R. Villagrán. **Distinguishing graphs by their spectra, Smith normal forms and complements.** arXiv preprint arXiv:2304.07217
- C.A. Alfaro & O. Zapata **The degree-distance and transmission-adjacency matrices.** arXiv preprint arXiv:2212.05297

Thank you

Carlos A. Alfaro
alfaromontufar@gmail.com