The Smith normal form of distance matrices of high dimensional trees

27th Conference of the ILAS 2026

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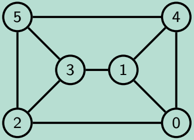
joint with J.U. Medrano and I. Téllez-Téllez

Distance matrix

Definition

Let G be a connected graph with n vertices.
The distance $d_G(u,v)$ between vertices u and v is the number of edges in a minimal walk between u and v.
The distance matrix $D(G)$ of G is the $n \times n$ matrix whose (u,v)-entry is $d_G(u,v)$.

Example



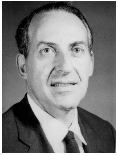
G

$$D(G) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Determinant of the distance matrix

Ron Graham

László Lovász



Henry O. Pollak

The celabrated formula of Graham, Lovász y Pollak states that

$$\det(D(T_{n+1})) = (-1)^n n 2^{n-1}$$

for any tree T_{n+1} with $n+1$ vertices.

The Smith normal form

Yaoping Hou

Ching Wah Woo

Hou y Woo extended Graham-Lovász-Pollak formula to

$$\mathrm{SNF}(D(T_{n+1})) = \mathrm{I}_2 \oplus 2\mathrm{I}_{\mathrm{n}-2} \oplus (2\mathrm{n})$$

for any tree T_{n+1} with $n+1$ vertices.

What is the Smith normal form?

Two integer matrices M and N are **equivalent** if there exist unimodular integer matrices P and Q such that $M = PNQ$.

The **Smith normal form** $\mathsf{SNF}(M)$ of an integer matrix M is the unique diagonal matrix $\mathsf{diag}(f_1, \ldots, f_r, 0, \ldots, 0)$ equivalent to M such that $r = range(M)$ and $f_i | f_j$ for $i < j$.

The **invariant factors** (o **elementary divisors**) of M are the integers in the diagonal of $\mathsf{SNF}(M)$.

A motivation

Theorem (A. Hoekstra-Mendoza, Serrano & Villagrán, 2025)

The 3-minors of the distance matrix of any connected bipartite graph are even numbers.

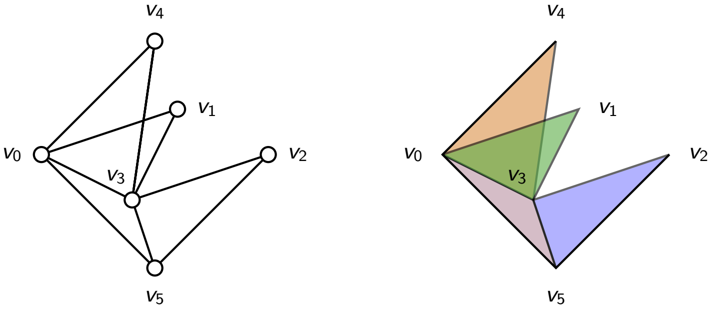
Corolary

The determinant of the distance matrix of a connected bipartite graph is even.

k-trees

A *k-clique* is a complete subgraph on k vertices.

The concept of k-trees may be defined recursively: a k-**tree** is either a complete graph on k vertices or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique.



These concepts can be analogously defined in terms of simplicial complexes.

Walks and distances in k-trees

Let T be a k-tree and $d \in \{1, \ldots, k\}$.

Two d-cliques τ and τ' in T are **adjacent** if they belong to the same $(d+1)$-clique σ, in such situation τ and τ' are **incident** to σ.

In this way, if τ and τ' are d-cliques, a d-**walk** between τ and τ' is a finite sequence $\tau_1\sigma_1\tau_2\sigma_2\cdots\tau_l$, where $\tau = \tau_1$, $\tau' = \tau_l$, and τ_i and τ_{i+1} are incident to the same $(d+1)$-clique σ_i.

Therefore, the d-**distance** from the d-cliques τ and τ' is the number of $(d+1)$-cliques in a minimum d-walk from τ and τ', and is denoted by $\text{dist}^d(\tau, \tau')$.

Note that there exists a d-walk between any pair of d-cliques in any k-tree T.

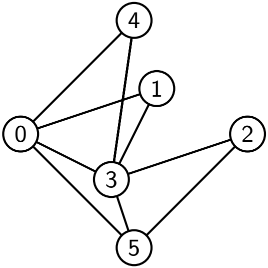
The d-distance matrix of a k-tree

Let $\mathbf{c_d}$ denote the number of d-cliques in T.

The d-**distance matrix** $\mathrm{D}^d(T)$ of the k-tree T is the $c_d \times c_d$ matrix, indexed by the d-cliques of T, such that

$$\mathrm{D}^d(T)_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \mathrm{dist}^d(\sigma_i, \sigma_j) & \text{otherwise.} \end{cases}$$

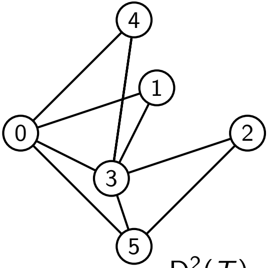
First distance matrix



$$D^1(T) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Note that $D^1(T) = D(T)$.

Second distance matrix



$$
D^2(T) = \begin{array}{c c} & \begin{array}{ccccccccc} 01 & 03 & 04 & 05 & 13 & 23 & 25 & 34 & 35 \end{array} \\ \begin{array}{c} 01 \\ 03 \\ 04 \\ 05 \\ 13 \\ 23 \\ 25 \\ 34 \\ 35 \end{array} & \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 3 & 3 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 2 & 2 & 3 & 3 & 1 & 2 \\ 2 & 1 & 2 & 0 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 0 & 3 & 3 & 2 & 2 \\ 3 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 1 \\ 3 & 2 & 3 & 2 & 3 & 1 & 0 & 3 & 1 \\ 2 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}
$$

Results

Theorem (Alfaro-Medrano-Téllez,2026)

Let $k \geq 1$ and $n \geq k+2$. For any k-tree T_n with n vertices,

$$\mathrm{SNF}(\mathrm{D}^k(T_n)) = \mathrm{I}_{(k-1)(n-k)+2} \oplus (k+1)\mathrm{I}_{n-k-2} \oplus [k(k+1)(n-k)].$$

Corolary (Alfaro-Medrano-Téllez,2026)

$\det(\mathrm{D}^k(T_n)) = (-1)^{k(n-k)} k(k+1)^{n-k-1}(n-k).$

Idea of the proof

- $$\mathrm{D}^k(T_n) \sim \begin{bmatrix} 0 & \mathbf{1}^T & \mathbf{1}^T & \cdots & \mathbf{1}^T \\ \mathbf{1} & -\mathrm{J}_k - \mathrm{I}_k & \mathbf{0}_{k,k} & \cdots & \mathbf{0}_{k,k} \\ \mathbf{1} & \mathbf{0}_{k,k} & -\mathrm{J}_k - \mathrm{I}_k & \cdots & \mathbf{0}_{k,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{0}_{k,k} & \mathbf{0}_{k,k} & \cdots & -\mathrm{J}_k - \mathrm{I}_k \end{bmatrix}.$$
- $\mathrm{SNF}(-\mathrm{J}_k - \mathrm{I}_k) = \mathrm{diag}(1, \dots, 1, k+1)$.
- $\mathrm{D}^k(T_n) \sim \mathrm{I}_{(k-1)(n-k)} \oplus \begin{bmatrix} n-k & 1^T \\ 1 & \mathrm{diag}(k+1, \dots, k+1) \end{bmatrix}.$
- $\mathrm{SNF} \begin{bmatrix} n-k & 1^T \\ 1 & \mathrm{diag}(k+1, \dots, k+1) \end{bmatrix} =$
  $\mathrm{diag}(1, 1, k+1, \dots, k+1, k(k+1)(n-k))$.

It could be possible to obtain a formula for the Smith normal form of the d-distance matrices of the k-trees with $d < k$?

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Thank you!

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