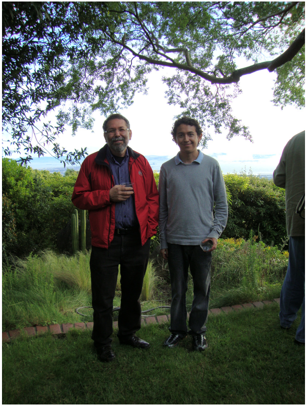
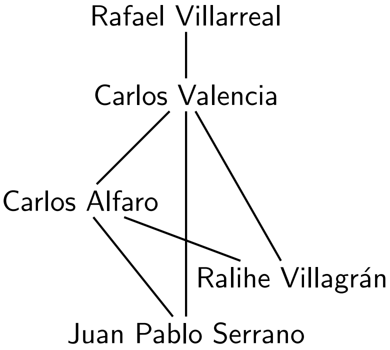
Distance ideals of graphs

75th Birthday of Rafael H. Villarreal

Carlos Alejandro Alfaro Montúfar
BANXICO



At David Eisenbud' home
Joint meeting AMS-SMM 2009



Math-Genealogy

Content

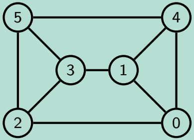
Properties of distance ideals

Distance matrix

Definition

Let G be a connected graph with n vertices.
The distance $d_G(u, v)$ between the vertices u and v is the number of edges of a shortest path between them.
The distance matrix $D(G)$ of G is the matrix with rows and columns indexed by the vertices of G where the uv-entry is equal to $d_G(u, v)$.

Example



G

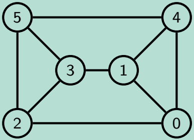
$$D(G) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The D_X matrix

Definition

Let $V(G) = \{v_0, \ldots, v_{n-1}\}$ be the vertex set.
The variables $X_G = \{x_0, \ldots, x_{n-1}\}$ are associated with $V(G)$.
Let $D_X(G) = diag(x_0, \ldots, x_{n-1}) + D(G)$.

Example



G

$$D_X(G) = \begin{bmatrix} x_0 & 1 & 1 & 2 & 1 & 2 \\ 1 & x_1 & 2 & 1 & 1 & 2 \\ 1 & 2 & x_2 & 1 & 2 & 1 \\ 2 & 1 & 1 & x_3 & 2 & 1 \\ 1 & 1 & 2 & 2 & x_4 & 1 \\ 2 & 2 & 1 & 1 & 1 & x_5 \end{bmatrix}$$

Distance ideals

Definition

Let $\mathfrak{R}[X]$ be the polynomial ring over a commutative ring $\mathfrak{R}$ with unity in the variables X.

For $1 \leq k \leq n$.

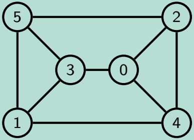
Let $\mathrm{minors}_k(D_X(G))$ be the set of determinants of the submatrices of size $k \times k$ of $D_X(G)$.

The *k-th* distance ideal $I_k^{\mathfrak{R}}(G)$ of G is the ideal $\langle \mathrm{minors}_k(D_X(G)) \rangle$ over $\mathfrak{R}[X]$.

An ideal is said to be unit or trivial if it is equal to $\langle 1 \rangle$.

Distance ideals of the prism with 6 vertices

Example



G

$$\begin{bmatrix} x_0 & 2 & 1 & 1 & 1 & 2 \\ 2 & x_1 & 2 & 1 & 1 & 1 \\ 1 & 2 & x_2 & 2 & 1 & 1 \\ 1 & 1 & 2 & x_3 & 2 & 1 \\ 1 & 1 & 1 & 2 & x_4 & 2 \\ 2 & 1 & 1 & 1 & 2 & x_5 \end{bmatrix}$$

The first three distance ideals are trivial.

A Gröbner bases of $I_4^{\mathbb{Z}}(G)$ is generated by the following polynomials:

$$x_0 + x_3 - 7, x_1 + x_4 - 7, x_2 + x_5 - 7, x_3x_4 - 2x_3 - 2x_4 + 7,$$
$$x_3x_5 - 5x_3 - 2x_5 + 7, 3x_3 - 3x_5, x_4x_5 - 2x_4 - 2x_5 + 7,$$
$$3x_4 + 3x_5 - 21, 3x_5^2 - 21x_5 + 21$$

Note $I_n^{\mathcal{R}}(G) = \langle \det(D_X(G)) \rangle$.

Varieties associated with distance ideals

Definition

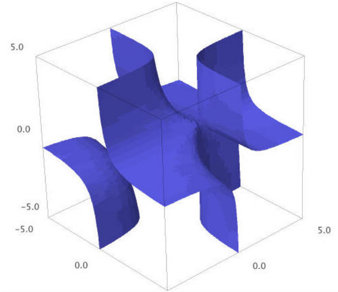
The variety $V(I)$ of the ideal I is the set of common roots of the polynomials in I.

Consider the complete graph K_3 with 3 vertices.

$I_1^{\mathbb{R}}(K_3) = \langle 1 \rangle \Rightarrow V(I_1^{\mathbb{R}}(K_3)) = \emptyset$.

$I_2^{\mathbb{R}}(K_3) = \langle x_0 - 1, x_1 - 1, x_2 - 1 \rangle \Rightarrow V(I_2^{\mathbb{R}}(K_3)) = \{(1,1,1)\}$.

$I_3^{\mathbb{R}}(K_3) = \langle x_0x_1x_2 - x_0 - x_1 - x_2 - 2 \rangle$.



Chains of distance ideals and its varieties

We have that

$$\langle 1\rangle \supseteq I_1^{\mathcal{R}}(G) \supseteq \cdots \supseteq I_n^{\mathcal{R}}(G) \supseteq \langle 0\rangle.$$

From which follows

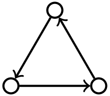
$$V(\langle 1\rangle) \subseteq V(I_1^{\mathcal{R}}(G)) \subseteq \cdots \subseteq V(I_n^{\mathcal{R}}(G)) \subseteq V(\langle 0\rangle).$$

Varieties of distance ideals generalize the spectrum of distance matrices.

Theorem (Abiad, A., Heysse & Vargas, 2022)

Let G and H two graphs on n vertices. Then, G and H are isomorphic if and only if there exists a permutation σ of $V(H)$ such that $I_n(G) = I_n(\sigma H)$.

Distance ideals of the circuit $\overrightarrow{C_3}$



$$D_X\left(\overrightarrow{C_3}\right) = \begin{bmatrix} x_0 & 1 & 2 \\ 2 & x_1 & 1 \\ 1 & 2 & x_2 \end{bmatrix}$$

Then $I_1^{\mathbb{Z}}\left(\overrightarrow{C_3}\right) = \langle 1 \rangle$ y $I_2^{\mathbb{Z}}\left(\overrightarrow{C_3}\right) = \langle x_0+3, x_1+3, x_2+3, 7\rangle$.
Meanwhile

$$I_3\left(\overrightarrow{C_3}\right) = \left\langle \det D_X\left(\overrightarrow{C_3}\right)\right\rangle = \langle x_0x_1x_2 - 2x_0 - 2x_1 - 2x_2 + 9\rangle.$$

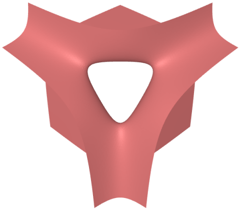


Figura: Partial view of the variety associated with $I_3\left(\overrightarrow{C_3}\right)$ in $\mathbb{R}^3$.

Distance ideals and the Smith normal form

Smith normal form

Two integer matrices M and N are **equivalent** if there exist unimodular integer matrices P and Q such that $M = PNQ$.

The **Smith normal form** $\mathrm{SNF}(M)$ of an integer matrix M is the unique diagonal matrix $\mathrm{diag}(f_1, \ldots, f_r, 0, \ldots, 0)$ equivalent to M such that $r = range(M)$ and $f_i | f_j$ for $i < j$.

The **invariant factors** (o **elementary divisors**) of M are the integers in the diagonal of $\mathrm{SNF}(M)$.

Theorem (Elementary divisors)

Let M be an integer matrix with $f_1, \ldots, f_r, 0, \ldots, 0$ its invariant factors. For $1 \leq k \leq r$, let Δ_k the gcd *of the k-minors of M, and $\Delta_0 = 1$. Then $f_k = \frac{\Delta_k}{\Delta_{k-1}}$.*

Distance ideals and the SNF

Proposition

Let $\mathbf{d} \in \mathbb{Z}^{V(G)}$ *and* $M := D_X(G)|_{X=\mathbf{d}}$. *If* $f_1 \mid \cdots \mid f_r$ *are the positive invariant factors of* M, *then*

$$I_k(G)|_{X=\mathbf{d}} = \langle \Delta_k(M) \rangle = \left\langle \prod_{j=1}^{k} f_j(M) \right\rangle,$$

for $k \in \{1, \ldots, r\}$, *where* r *is the rank of* M.

Then, evaluating the distance ideals ($\mathbb{Z}[X]$) at $X = \mathbf{0}$, $trs(G)$, $-trs(G)$, $\deg(G)$ or $-\deg(G)$, we can recover the SNF of the matrices D, D^Q, D^L, D_+^{deg} o D^{deg}, respectively.

Distance ideals and the SNF

Example

$$I_k^{\mathbb{Z}}(K_3) = \begin{cases} \langle 1 \rangle & \text{if } k = 1, \\ \langle x_0 - 1, x_1 - 1, x_2 - 1 \rangle & \text{if } k = 2, \\ \langle x_0 x_1 x_2 - x_0 - x_1 - x_2 + 2 \rangle & \text{if } k = 3. \end{cases}$$

- Evaluating at $X = \mathbf{0}$
  $$I_k^{\mathbb{Z}}(K_3)|_{X=\mathbf{0}} = \langle \Delta_i(D(K_3)) \rangle = \begin{cases} \langle 1 \rangle & \text{if } k = 1, \\ \langle 1 \rangle & \text{if } k = 2, \\ \langle 2 \rangle & \text{if } k = 3. \end{cases}$$
  Then $\text{SNF}(D(K_3)) = \text{diag}(1, 1, 2)$.
- Evaluating at $X = -(2, 2, 2)$, we get $\text{SNF}(D^L(G)) = \text{diag}(1, 3, 0)$
- Evaluating at $X = (2, 2, 2)$, we get $\text{SNF}(D^Q(G)) = \text{diag}(1, 1, 4)$

Distance ideals of complete graphs

There are few families of graphs for which thier distance ideals are known.

Theorem (Corrales & Valencia, 2013)

The k-th distance ideal of K_n with n vertices is generated by:

$$\begin{cases} \prod_{j=1}^{n}(x_j - 1) + \sum_{i=1}^{n}\prod_{j\neq i}(x_j - 1) & \text{if } k = n, \\ \left\{\prod_{j\in\mathcal{I}}(x_j - 1) : \mathcal{I} \subset [n] \text{ and } |\mathcal{I}| = k-1\right\} & \text{if } k < n. \end{cases}$$

Corollary

- $\mathrm{SNF}(D(K_n)) = \mathsf{I}_{n-1} \oplus [n-1]$
- $\mathrm{SNF}(D^L(K_n)) = [1] \oplus n\mathsf{I}_{n-2} \oplus [0]$
- $\mathrm{SNF}(D^Q(K_n)) = [1] \oplus (n-2)\mathsf{I}_{n-2} \oplus [2(n-1)(n-2)]$

Distance ideals of the stars

Theorem (A. & Taylor, 2020)

Let $D_X(K_{m,1}) = \begin{bmatrix} \text{diag}(x_1,\ldots,x_m) - 2\mathsf{I}_m + 2\mathsf{J}_m & \mathsf{J}_{m,1} \\ \mathsf{J}_{1,m} & y \end{bmatrix}$.

Then $\det(D_X(K_{m,1})) = y\prod_{i=1}^{m}(x_i-2) + (2y-1)\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}(x_j-2)$.

For $k \in [m]$, *the* k*-th distance ideal of the star* $K_{m,1}$ *is generated by* $\langle E_k \cup F_k \rangle$. *Where* $E_k = \left\{ \prod_{i\in\mathcal{I}}(x_i-2) : \mathcal{I} \subset [m] \text{ and } |\mathcal{I}| = k-1 \right\}$ *and* $F_k = \left\{ (2y-1)\prod_{i\in\mathcal{I}}(x_i-2) : \mathcal{I} \subset [m] \text{ and } |\mathcal{I}| = k-2 \right\}$.

Corollary

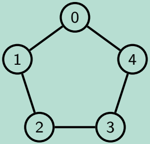
- $\text{SNF}(D(K_{m,1})) = \mathsf{I}_2 \oplus 2\mathsf{I}_{m-2} \oplus [2m]$
- $\text{SNF}(D^L(K_{m,1})) = [1] \oplus (2m+1)\mathsf{I}_{m-1} \oplus [0]$

Characterizations

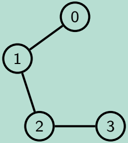
Distance ideals of induced subgraphs

Distance ideals DO NOT behave well by taking induced subgraphs.

Example



$$D(G) = \begin{bmatrix} 0 & 1 & 2 & \boxed{2} & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ \boxed{2} & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$



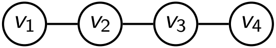
$$D(G) = \begin{bmatrix} 0 & 1 & 2 & \boxed{3} \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ \boxed{3} & 2 & 1 & 0 \end{bmatrix}$$

Distance ideals of induced subgraphs

Lemma

The second distance ideal of any graph containing P_4 as induced subgraph is trivial.

Proof. Let G be a graph containing P_4 as induced subgraph.

Suppose $P_4 =$ 

Then $D_X(G)$ contains the following submatrix:

$$M = D_X(G)[V(P_4); V(P_4)] = \begin{bmatrix} x_1 & 1 & 2 & 2 \\ \boxed{1} & x_2 & \boxed{1} & 2 \\ 2 & 1 & x_3 & 1 \\ \boxed{2} & 2 & \boxed{1} & x_4 \end{bmatrix}$$

Since $\det(M[\{v_2, v_4\}; \{v_1, v_3\}]) = -1$, then $I_2^{\mathcal{R}}(G) = \langle 1 \rangle$.

Distance ideals of induced subgraphs

Then, P_4 is **forbidden** for the graphs with only one trivial distance ideal.

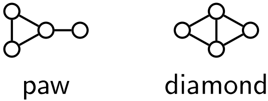
Could be characterized the graphs with only one trivial distance ideal?

Graphs with one trivial distance ideal

Theorem (A. & Taylor,2020)

The following statements are equivalent:

1. G *has ony one trivial distance ideal over* $\mathbb{Z}[X]$.
2. G *is* $\{P_4, \text{paw}, \text{diamond}\}$*-free.*
3. G *is either* $K_{m,n}$ *or* K_n.



Theorem (A. & Taylor,2020)

The following statements are equivalent:

1. G *has ony one trivial distance ideal over* $\mathbb{R}[X]$.
2. G *is* $\{P_4, \text{paw}, \text{diamond}, C_4\}$*-free.*
3. G *is* $K_{1,n}$ *ó* K_n.

Digraphs with one trivial distance ideal

Proposition

For $n \geq 5$, $I_2^{\mathbb{Z}}\left(\overrightarrow{C_n}\right)$ *is trivial and* $I_3^{\mathbb{Z}}\left(\overrightarrow{C_n}\right) = \langle x_1, x_2, \ldots, x_n, n \rangle$.

That is, there is an infinite number of minimal forbidden digraphs for the family of strong digraphs with only one trivial distance ideal.

Proposition

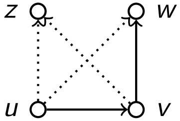
If G *is a strong digraph with* $\operatorname{diam}(G) \geq 3$, *then* $I_2^{\mathbb{Z}}(G) = \langle 1 \rangle$.

Then, the strong digraphs with only one trivial distance ideal have diameter at most 2.

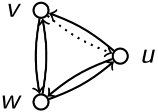
Digraphs with one trivial distance ideal

Definition

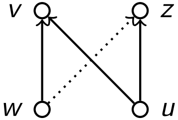
Let $D = (V, A)$ be a strong digraph, a **pattern** $\mathcal{P} = (U, B, C)$ in D is the subset of vertices $U \subseteq V$ together with two disjoint sets of arcs, B and C, whose endpoints are in U, where arcs in B are arcs in A while arcs in C must not belong to A.



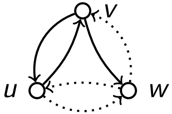
$\mathcal{F}_1$



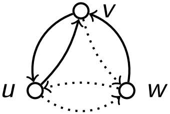
$\mathcal{F}_2$



$\mathcal{F}_3$



$\mathcal{F}_4$



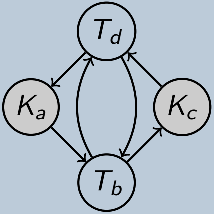
$\mathcal{F}_5$

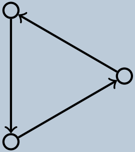
Digraphs with one trivial distance ideal

Theorem (A., Hoekstra, Serrano y Villagrán, 2025)

Let G be a strong digraph. The following statements are equivalent:

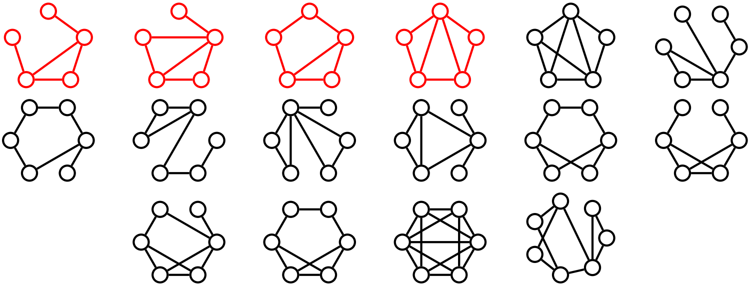
1. *G has only one trivial distance ideal over $\mathbb{Z}[X]$,*
2. *The patterns $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ are forbidden for G,*
3. *G is one the strong digraphs described by the following diagrams:*





where T_p denote an independent set of order $p \geq 0$, y K_q denote a complete graph of order $q \geq 0$, and an arc between two sets, say A and B, means that there exists an arc from each vertex in A to each vertex in B.

Graphs with two trivial distance ideals



Family **F** of graphs.

Theorem (A., 2020)

The graphs with 2 trivial distance ideals over $\mathbb{Z}[X]$ are free of the 16 graphs in the figure and the cycles of odd length greater or equal than 7.

$\{F, \text{odd-holes}_7\}$-free graphs are related with **distance-hereditary** graphs, **perfect** graphs and **trees**, among others.

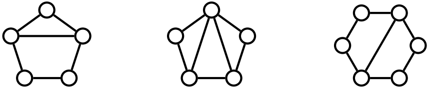
Distance-hereditary graphs

A graph is distance-hereditary if for each induced subgraph H of G, and each pair of vertices $u, v \in V(H)$, $d_H(u, v) = d_G(u, v)$.

Ed Howorka

Distance-hereditary graphs:

- were introduced by Howorka in 1977.
- are characterized by the graph free of **house**, **domino**, **gem** or cycles of length 5 or greater.



house **gem** **domino**

E. Howorka, *A characterization of distance-hereditary graphs.* **Quart. J. Math. Oxford Ser.** 28 (1977) 112 417–420.

Perfect graphs

Properties of distance-hereditary graphs:

- are **perfect** graphs, that is, the chromatic number equals the clique number of each subgraph.

Theorem (Strong perfect graphs, Chudnovsky, et. al., 2006)

A graph G is perfect if and only if G and $\overline{G}$ do **not** *contain a cycle of odd length greater or equal than 5.*

M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, *The strong perfect graph theorem*, **Ann. Math.** 164 (1) (2006) 51–229.

Trees

Hou

Woo

Yaoping Hou and Ching Wah Woo computed the SNF of the distance matrix of trees.
In particular, they proved that the distance matrix of trees have exactly 2 invariant factors equal to 1. Therefore,

$$\text{trees} \subseteq \{F, \text{odd-holes}_7\}\text{-free graphs.}$$

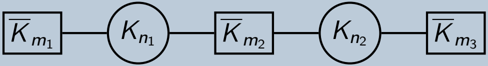
Y. Hou, Yaoping & C. Woo. *Distance unimodular equivalence of graphs.* **Linear Multilinear Algebra** 56 (2008) 611–626.

Graphs with 2 trivial distance ideals

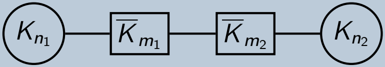
Theorem (A. Hoekstra-Mendoza, Serrano & Villagrán, 2025)

Let G be a simple connected graph. The following are equivalent:

1. *G has 2 trivial distance ideals over $\mathbb{Z}[X]$.*
2. *G is $\{\mathcal{F}, \text{odd-holes}_7\}$-free.*
3. *G is one of the following graphs:*
   - i) C_5,
   - ii) *a connected bipartite graph,*
   - iii) *a tripartite complete graph,*
   - iv) $K_{n-p+1,1,\ldots,1}$ *where p is the number of partitions,*
   - v) *is an induced subgraph of*



   - vi) *or an induced subgraph of*

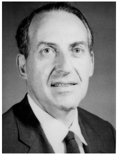


Trees

Other consequences

Ron Graham

László Lovász



Henry O. Pollak

The celebrated Graham, Lovász y Pollak formula states that

$$\det(D(T_{n+1})) = (-1)^n n 2^{n-1}$$

for any tree T_{n+1} with $n+1$ vertices.
Hou y Woo extended the Graham-Lovász-Pollak result to obtain the SNF of the distance matrix for any tree T_{n+1} with $n+1$ vertices, proving that

$$\mathrm{SNF}(D(T_{n+1})) = \mathrm{I}_2 \oplus 2\mathrm{I}_{\mathrm{n}-2} \oplus (2\mathrm{n}).$$

R. Graham & H.O. Pollak. *On the addressing problem for loop switching*. **Bell System Tech. J.** 50 (1971) 2495–2519.
R. Graham & L. Lovász. *Distance matrix polynomials of trees.* **Adv. in Math.** 29 (1978) 60–88.

Other consequences

Theorem (A. Hoekstra-Mendoza, Serrano & Villagrán, 2025)

Every 3-minor of the distance matrix of a connected bipartite graph is an even number.

Corollary

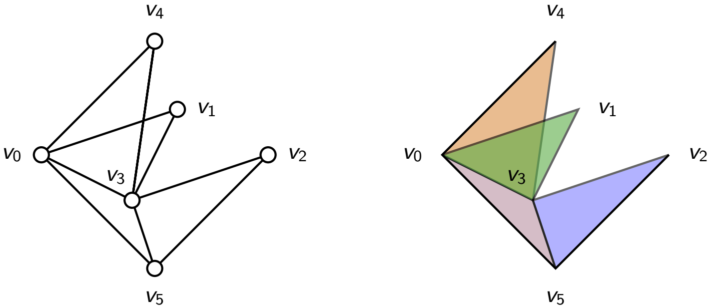
The determinant of the distance matrix of a connected bipartite graph is an even number.

Could be possible to obtain the SNF of the distance matrix of any connected bipartite graph?

k-trees

A k-**clique** is a complete subgraph on k vertices.

The concept of k-trees may be defined recursively: a k-**tree** is either a complete graph on k vertices or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique.



These concepts can be analogously defined in terms of simplicial complexes.

Walks and distances in k-trees

Let T be a k-tree and $d \in \{1, \ldots, k\}$.

Two d-cliques τ and τ' in T are **adjacent** if they belong to the same $(d+1)$-clique σ, in such situation τ and τ' are **incident** to σ.

In this way, if τ and τ' are d-cliques, a d-**walk** between τ and τ' is a finite sequence $\tau_1\sigma_1\tau_2\sigma_2\cdots\tau_l$, where $\tau = \tau_1$, $\tau' = \tau_l$, and τ_i and τ_{i+1} are incident to the same $(d+1)$-clique σ_i.

Therefore, the d-**distance** from the d-cliques τ and τ' is the number of $(d+1)$-cliques in a minimum d-walk from τ and τ', and is denoted by $\text{dist}^d(\tau, \tau')$.

Note that there exists a d-walk between any pair of d-cliques in any k-tree T.

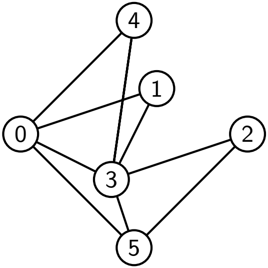
The d-distance matrix of a k-tree

Let $\mathbf{c_d}$ denote the number of d-cliques in T.

The d-**distance matrix** $\mathrm{D}^d(T)$ of the k-tree T is the $c_d \times c_d$ matrix, indexed by the d-cliques of T, such that

$$\mathrm{D}^d(T)_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \mathrm{dist}^d(\sigma_i, \sigma_j) & \text{otherwise.} \end{cases}$$

First distance matrix

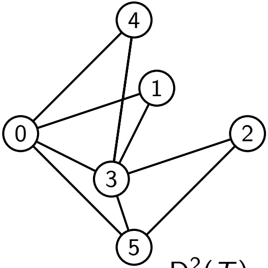


$D^1(T) =$

	0	1	2	3	4	5
0	0	1	2	1	1	1
1	1	0	2	1	2	2
2	2	2	0	1	2	1
3	1	1	1	0	1	1
4	1	2	2	1	0	2
5	1	2	1	1	2	0

Note that $D^1(T) = D(T)$.

Second distance matrix



$D^2(T) =$

	01	03	04	05	13	23	25	34	35
01	0	1	2	2	1	3	3	2	2
03	1	0	1	1	1	2	2	1	1
04	2	1	0	2	2	3	3	1	2
05	2	1	2	0	2	2	2	2	1
13	1	1	2	2	0	3	3	2	2
23	3	2	3	2	3	0	1	3	1
25	3	2	3	2	3	1	0	3	1
34	2	1	1	2	2	3	3	0	2
35	2	1	2	1	2	1	1	2	0

The SNF of D^k of the k-trees

Theorem (Alfaro-Medrano-Téllez,2026)

Let $k \geq 1$ and $n \geq k+2$. For any k-tree T_n with n vertices,

$$\mathrm{SNF}(\mathrm{D}^k(T_n)) = \mathrm{I}_{(k-1)(n-k)+2} \oplus (k+1)\mathrm{I}_{n-k-2} \oplus [k(k+1)(n-k)].$$

Corollary (Alfaro-Medrano-Téllez,2026)

$\det(\mathrm{D}^k(T_n)) = (-1)^{k(n-k)} k(k+1)^{n-k-1}(n-k).$

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Thank you! and
Happy birthday Rafael!

Carlos A. Alfaro

https://alfaromontufar.github.io
alfaromontufar@gmail.com