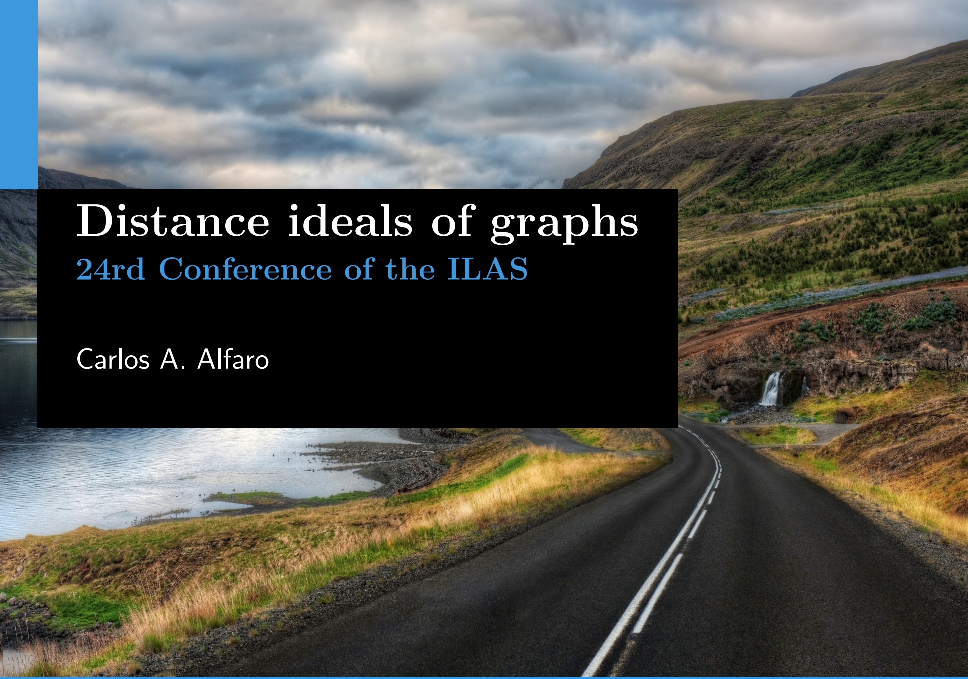


Distance ideals of graphs

24rd Conference of the ILAS

Carlos A. Alfaro

This work is co-authored with:

Aida Abiad
Ghent University
Belgium

Kristin Heysse
Macalester College
USA

Libby Taylor
Stanford University
USA

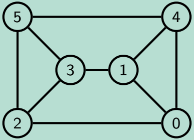
Marcos C. Vargas
Banco de México
México

Distance matrix of graphs

Definition

Given a connected graph $\mathbf{G}$ with n vertices. The distance matrix $\mathbf{D}(\mathbf{G})$ of G is the $n \times n$ matrix whose uv-entry is the distance $\mathbf{d_G}(\mathbf{u}, \mathbf{v})$ between the vertices u and v.

Example



G

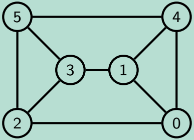
$$\begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Distance ideals of graphs

Definition

Given a graph G with vertex set $v_0, \ldots, v_{n-1}$.
Let $D_X(G) = diag(x_0, \ldots, x_{n-1}) - D(G)$, where $x_0, \ldots, x_{n-1}$ are indeterminates.

Example



G

$$D_X(G) = \begin{bmatrix} x_0 & 1 & 1 & 2 & 1 & 2 \\ 1 & x_1 & 2 & 1 & 1 & 2 \\ 1 & 2 & x_2 & 1 & 2 & 1 \\ 2 & 1 & 1 & x_3 & 2 & 1 \\ 1 & 1 & 2 & 2 & x_4 & 1 \\ 2 & 2 & 1 & 1 & 1 & x_5 \end{bmatrix}$$

Distance ideals of graphs

Definition

Let $\mathcal{R}[X_G]$ denote the polynomial ring over a commutative ring $\mathcal{R}$ in the variables X_G.

Let $\mathrm{minors}_k(D_X(G))$ be the set of determinants (polynomials) of the $k \times k$ submatrices of $D_X(G)$.

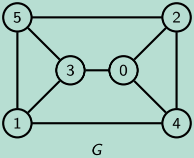
For $1 \leq k \leq n$ the k-th distance ideal $D_k^{\mathcal{R}}(G, X_G)$ is the ideal $\langle \mathrm{minors}_k(D(G, X_G)) \rangle$.

An ideal is said to be trivial if it is equal to $\langle 1 \rangle$ $(= \mathcal{R}[X_G])$.

Let $\Phi_{\mathcal{R}}(G)$ be the maximum integer k for which $D_k^{\mathcal{R}}(G, X)$ is trivial.

Distance ideals of graphs

Example



$$\begin{bmatrix} x_0 & 2 & 1 & 1 & 1 & 2 \\ 2 & x_1 & 2 & 1 & 1 & 1 \\ 1 & 2 & x_2 & 2 & 1 & 1 \\ 1 & 1 & 2 & x_3 & 2 & 1 \\ 1 & 1 & 1 & 2 & x_4 & 2 \\ 2 & 1 & 1 & 1 & 2 & x_5 \end{bmatrix}$$

$\Phi_{\mathbb{Z}}(G) = 3$

A Gröbner basis for $D_4^{\mathbb{Z}}(G, X)$ es generated by the following polynomials:

$$x_0 + x_3 - 7, x_1 + x_4 - 7, x_2 + x_5 - 7, x_3x_4 - 2x_3 - 2x_4 + 7,$$
$$x_3x_5 - 5x_3 - 2x_5 + 7, 3x_3 - 3x_5, x_4x_5 - 2x_4 - 2x_5 + 7,$$
$$3x_4 + 3x_5 - 21, 3x_5^2 - 21x_5 + 21$$

Note $D_n^{\mathcal{R}}(G, X) = \langle \det(D_X(G)) \rangle$.

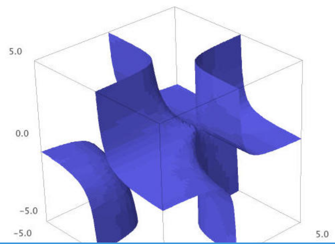
Distance ideals of graphs

Definition

The variety $V(I)$ of an ideal I is the set of common roots between polynomials in I.

Example

Consider the complete graph K_3 with 3 vertices.
$\Phi_{\mathbb{R}}(K_3) = 1$,
$D_2^{\mathbb{R}}(K_3, X) = \langle x_0 - 1, x_1 - 1, x_2 - 1\rangle$, whose $V(D_2^{\mathbb{R}}(K_3, X)) = \{(1,1,1)\}$
$I_3^{\mathbb{R}}(K_3, X_{K_3}) = \langle x_0x_1x_2 - x_0 - x_1 - x_2 - 2\rangle$ and $V(I_3^{\mathbb{R}}(K_3, X_{K_3}))$



Distance ideals of graphs

We have that

$$\langle 1\rangle \supseteq D_1^{\mathcal{R}}(G,X) \supseteq \cdots \supseteq D_n^{\mathcal{R}}(G,X) \supseteq \langle 0\rangle.$$

Then

$$V(\langle 1\rangle) \subseteq V(D_1^{\mathcal{R}}(G,X)) \subseteq \cdots \subseteq V(D_n^{\mathcal{R}}(G,X)) \subseteq V(\langle 0\rangle).$$

Some observations

- The varieties of $D(G,X)$ generalize the spectrum of D, D^L y D^Q,
- By evaluating distance ideals (over $\mathbb{Z}[X]$) at $X=\mathbf{0}$ or $X=Tr(G)$, we can recover the SNF of D, D^L y D^Q.

Proposition (A. & Taylor, 2020)

Evaluating $D_k^{\mathbb{Z}}(G,X)$

- *at $X=\mathbf{0}$, we obtain an ideal generated by $\Delta_k(D(G))$.*
- *at $X=-Tr(G)$, we obtain an ideal generated by $\Delta_k(D^L(G))$.*
- *at $X=Tr(G)$, we obtain an ideal generated by $\Delta_k(D^Q(G))$.*

Primeras observaciones

Example

$$D_k^{\mathbb{Z}}(K_3, X) = \begin{cases} \langle 1 \rangle & \text{si } k = 1, \\ \langle x_0 - 1, x_1 - 1, x_2 - 1 \rangle & \text{si } k = 2, \\ \langle x_0x_1x_2 - x_0 - x_1 - x_2 + 2 \rangle & \text{si } k = 3. \end{cases}$$

- Evaluating at $X = \mathbf{0}$

$$D_i^{\mathbb{Z}}(K_3, X)|_{X=\mathbf{0}} = \langle \Delta_i(D(K_3)) \rangle = \begin{cases} \langle 1 \rangle & \text{si } k = 1, \\ \langle 1 \rangle & \text{si } k = 2, \\ \langle 2 \rangle & \text{si } k = 3. \end{cases}$$

  then $SNF(D(K_3)) = \text{diag}(1, 1, 2)$.
- Evaluating at $X = (-2, -2, -2)$, then $SNF(D^L(G)) = \text{diag}(1, 3, 0)$
- Evaluating at $X = (2, 2, 2)$, then $SNF(D^Q(G)) = \text{diag}(1, 1, 4)$

Codeterminantal graphs

Theorem

Let G and G' be two graphs with n vertices. Then G and G' are isomorphic if and only if there exists a permutation σ on V such that $det(D_X(G)) = det(D_{\sigma X}(\sigma G'))$.

Definition

Two graphs G and H are $M_x^{\mathcal{R}}$-*codeterminantal* if $I_k^{\mathcal{R}}(M_x(G)) = I_k^{\mathcal{R}}(M_x(H))$ for each $k \in [n]$.

Definition

Two graphs G and H are $M^{\mathcal{R}}$-*codeterminantal* if $I_k^{\mathcal{R}}(M(G)) = I_k^{\mathcal{R}}(M(H))$ for each $k \in [n]$.

Codeterminantal graphs

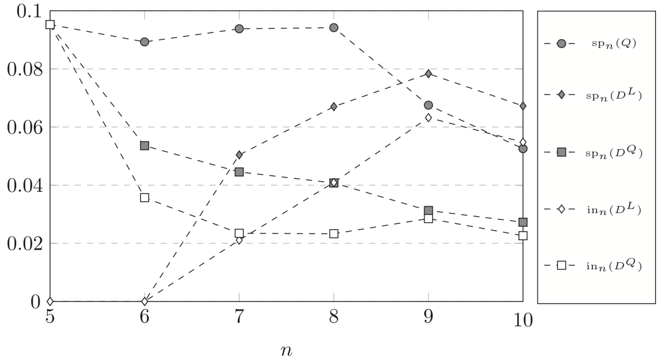


Figure: $sp(M)$ denotes the fraction of graph with n vertices having a M-coespectral mate. $in(M)$ denotes the fraction of graph with n vertices having a M-coinvariant mate.

Main references

- C.A. Alfaro & L. Taylor, *Distance ideals of graphs.* **Linear Algebra Appl.** 584 (2020) 127–144.
- C.A. Alfaro, *On graphs with two trivial distance ideals.* **Linear Algebra Appl.** 597 (2020) 69–85.
- A. Abiad & C.A. Alfaro, *Enumeration of cospectral and coinvariant graphs.* **Appl. Math. Comput.** 408 (2021) 126348.
- A. Abiad, C.A. Alfaro, K. Heysse & M. Vargas, *Codeterminantal graphs*. To appear in **Linear Algebra Appl.**

Thank you

Carlos A. Alfaro
alfaromontufar@gmail.com